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STRESSES AROUND A CIRCULAR HOLE IN A SHALLOW CONICAL SHELL

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Abstract—Analytical solutions are presented for the stresses in a shallow conical shell having a circular hole on its lateral surface. The shell which is closed at both ends, is subjected to a uniform axial tension and internal pressure balanced only by distributed transverse shear forces at the boundary of the hole. The method of analysis involves perturbations in parameters defining curvature and the cone angle of the shell (β and ε respectively). For small values of these parameters, significant membrane and bending stresses are obtained retaining terms of order of β^2 and ε^2.

NOTATION

ν	Poisson's ratio
E	Young's modulus
h	thickness of the shell
R_0	second principal radius of curvature in the circumferential direction at the centre of the hole (Fig. 1)
α	semicone angle
r_0	radius of the hole
D	$\dfrac{Eh^3}{12(1-\nu^2)}$, bending rigidity
m	$\dfrac{\sqrt{[12(1-\nu^2)]}}{Eh^2}$, a constant
β	$\dfrac{[3(1-\nu^2)]^{\frac{1}{4}}}{2}\dfrac{r_0}{(R_0h)^{\frac{1}{2}}}$ non-dimensional curvature parameter
ε	$\dfrac{r_0 \tan\alpha}{R_0}$; non-dimensional parameter
γ	Euler–Masceroni's constant ($\ln\gamma = 0{\cdot}5772$)
p	uniform internal pressure
P	total axial load
q_0	pR_0, internal pressure constant
p_0	$\dfrac{P}{2\pi R_0\cos^2\alpha}$, axial load constant
sr_0, ψ	polar coordinates with apex as the origin (Fig. 1a)
rr_0, θ	polar coordinates with the centre of the hole as the origin (Fig. 1a)
λ	angle between s and r (Fig. 1a)
$N_s, N_\psi, N_{s\psi}$	membrane forces in polar coordinates with apex as the origin
$N_r, N_\theta, N_{r\theta}$	membrane forces in polar coordinates with the centre of the hole as the origin (Fig. 2)
$M_r, M_\theta, M_{r\theta}$	stress couples in polar coordinates with the centre of the hole as the origin (Fig. 2)
Q_r, Q_r^*	transverse shear and effective transverse shear per unit length of shell element (Fig. 2)
W	normal displacement of a point on the middle surface of the shell, positive outwards
F	stress function

ϕ	complex stress–displacement function $(W-imF)$, $i=\sqrt{-1}$
$H_n^{(1)}[\sqrt{(2i)}\beta r]$	Hankel functions of first kind and nth order
superscripts	T corresponds to total solution: A bar ($^-$) corresponds to initial undisturbed stress system

1. INTRODUCTION

THE determination of the state of stress around circular openings in cylindrical shells has received considerable attention in recent years [1–3]. On the other hand, a systematic analysis of stresses around openings in conical shells is still not available. A first attempt at an approximate analysis of stresses around circular opening in conical shell was made

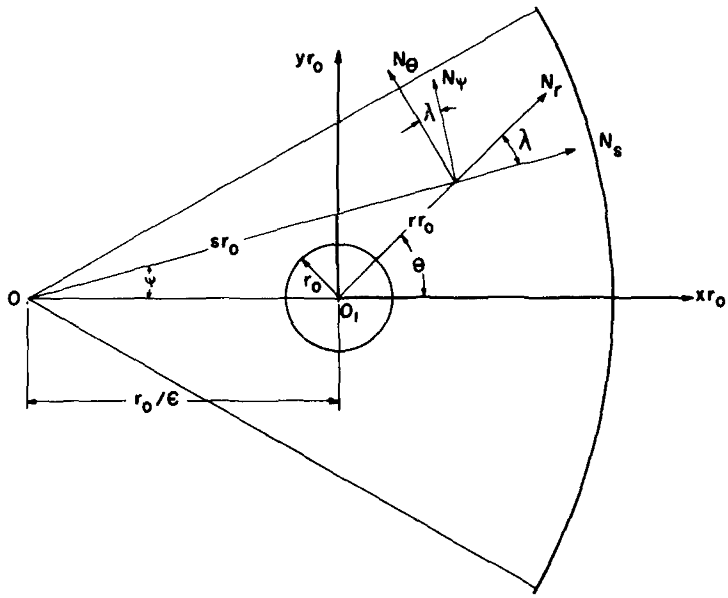

FIG. 1(a). Conical shell developed on a flat surface.

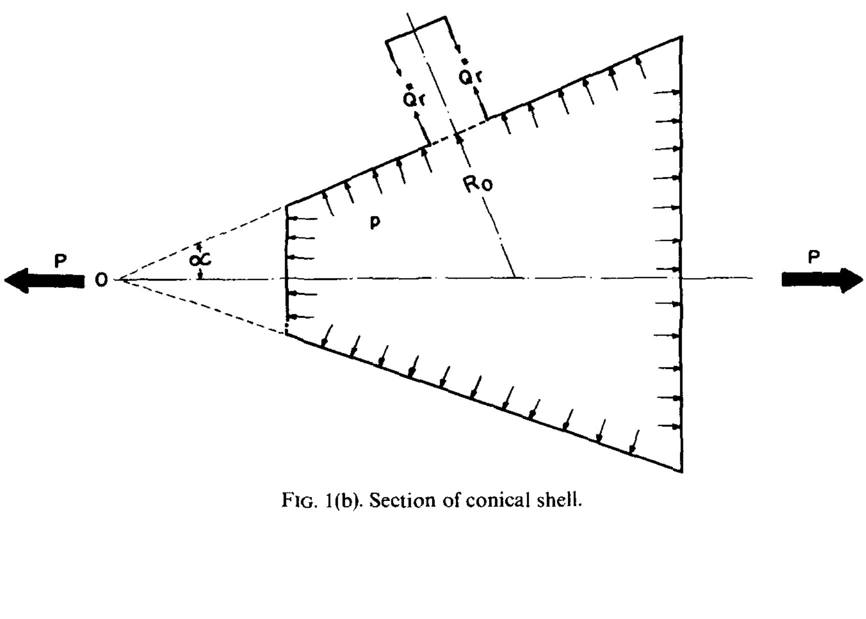

FIG. 1(b). Section of conical shell.

by Guz [4]. Perturbation method was used to determine the membrane state of stress due to axial and torsional loads. However, due to certain errors in the differential equations, these solutions are incorrect. These errors are eliminated in this paper and the work is extended to include the important case of internal pressure. Further, both the bending and membrane solutions are obtained for small values of the parameters defining the curvature and the cone angle (β and ε respectively) by perturbation method. Therefore, the present study is a first step to obtain a systematic solution for this problem. Formulae, from which the membrane and bending stresses can be computed, are presented and numerical results are given for various values of these parameters.

2. THE GOVERNING EQUATIONS

2.1 *The differential equation*

The differential equations for shallow thin shells are given in curvilinear co-ordinates in [5]. In these equations, if the non-dimensional co-ordinates s, ψ of a conical shell are substituted, the following two governing equations, involving the membrane stress function F and the normal displacement W, result

$$\nabla^4 W(s,\psi) + \frac{r_0}{Ds\tan\alpha}\frac{\partial^2 F(s,\psi)}{\partial s^2} = \frac{pr_0^4}{D} \tag{1a}$$

$$\nabla^4 F(s,\psi) - \frac{Ehr_0}{s\tan\alpha}\frac{\partial^2 W(s,\psi)}{\partial s^2} = 0 \tag{1b}$$

where

$$\nabla^4 = \left(\frac{\partial^2}{\partial s^2} + \frac{1}{s}\frac{\partial}{\partial s} + \frac{1}{s^2}\frac{\partial^2}{\partial \psi^2}\right)^2.$$

By defining a new function $\phi = W - imF$, the differential equations (1a) and (1b) reduce to a single equation

$$\nabla^4\phi + \frac{8i\beta^2}{\varepsilon s}\frac{\partial^2\phi}{\partial s^2} = \frac{pr_0^4}{D} \tag{2}$$

where

$$m = \frac{\sqrt{[12(1-\nu^2)]}}{Eh^2}$$

$$\beta = \frac{[3(1-\nu^2)]^{1/4}}{2}\frac{r_0}{(R_0 h)^{1/2}}, \quad \text{is a curvature parameter}$$

and

$$\varepsilon = \frac{r_0\tan\alpha}{R_0}, \quad \text{defines the cone angle.}$$

We now look upon this problem as a sum of the initial problem corresponding to the uniform stresses in the absence of hole and the residual problem corresponding to the additional stresses arising due to the hole. We denote the quantities corresponding to the initial problem by $\bar{\phi}$, $\bar{W}$, $\bar{F}$, $\bar{N}_r \ldots$ etc. and those corresponding to the total problem by ϕ^T, W^T, F^T, N_r^T, etc. We get the total solution as

$$\phi^T = \bar{\phi}+\phi, \qquad W^T = \bar{W}+W,$$
$$F^T = \bar{F}+F, \qquad N_r^T = \bar{N}_r + N_r, \quad \text{etc.}$$

Here, the symbols $\phi, W, F, N_r \ldots$ etc. correspond to the residual problem. For the residual problem, equation (2) reduces to a homogeneous equation

$$\nabla^4\phi+\frac{8i\beta^2}{\varepsilon s}\frac{\partial^2\phi}{\partial s^2} = 0. \tag{3}$$

By substituting,

$$s\sin\psi = y, \qquad s\cos\psi = \frac{1}{\varepsilon}+x,$$

equation (3) becomes

$$\nabla^4\phi+8i\beta^2\left[\frac{1}{\varepsilon s}\frac{\partial^2\phi}{\partial x^2}+\frac{2y(\varepsilon x+1)}{\varepsilon^2 s^3}\frac{\partial^2\phi}{\partial x\partial y}+\frac{y^2}{\varepsilon s^3}\left(\frac{\partial^2\phi}{\partial y^2}-\frac{\partial^2\phi}{\partial x^2}\right)\right] = 0. \tag{4}$$

2.2 *Stress resultants*

In (r, θ) polar co-ordinates, the stress resultants are determined by

$$\begin{aligned}
N_r &= \frac{1}{r_0^2}\left(\frac{1}{r}\frac{\partial F}{\partial r}+\frac{1}{r^2}\frac{\partial^2 F}{\partial\theta^2}\right)\\
N_\theta &= \frac{1}{r_0^2}\frac{\partial^2 F}{\partial r^2}\\
N_{\theta r} &= -\frac{1}{r_0^2}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial F}{\partial\theta}\right)\\
M_r &= \frac{D}{r_0^2}\left(\frac{\partial^2 W}{\partial r^2}+\frac{\nu}{r^2}\frac{\partial^2 W}{\partial\theta^2}+\frac{\nu}{r}\frac{\partial W}{\partial r}\right)\\
M_\theta &= \frac{D}{r_0^2}\left(\frac{1}{r^2}\frac{\partial^2 W}{\partial\theta^2}+\frac{1}{r}\frac{\partial W}{\partial r}+\nu\frac{\partial^2 W}{\partial r^2}\right)\\
M_{r\theta} = M_{\theta r} &= \frac{D(1-\nu)}{r_0^2}\left(\frac{1}{r}\frac{\partial^2 W}{\partial r\partial\theta}-\frac{1}{r^2}\frac{\partial W}{\partial\theta}\right)\\
Q_r^* &= \frac{D}{r_0^3}\left[\frac{\partial}{\partial r}\nabla^2 W+\frac{(1-\nu)}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial^2 W}{\partial\theta^2}\right)\right]
\end{aligned} \tag{5}$$

where Q_r^* is the effective transverse shear

$$Q_r^* = Q_r+\frac{1}{r}\frac{\partial M_{r\theta}}{\partial\theta}.$$

The positive stress resultants are shown on a shell element in Fig. 2.

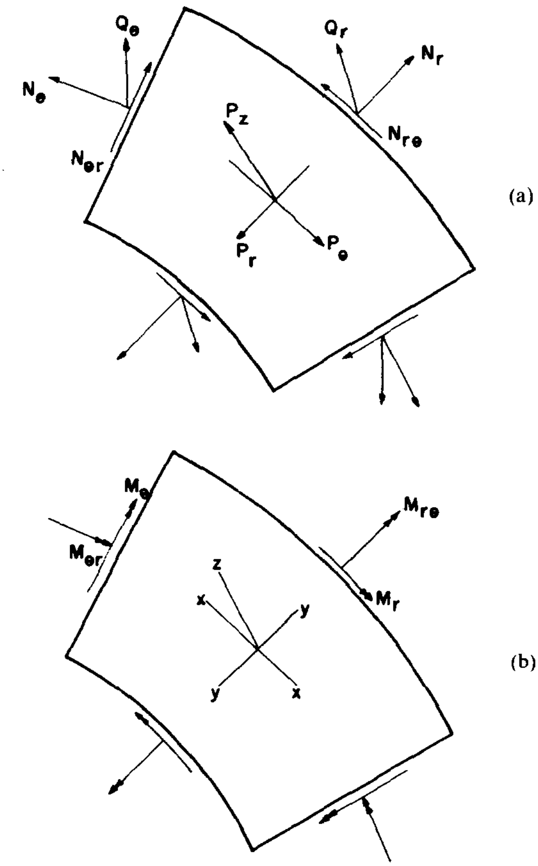

FIG. 2. Shell element: (a) stress resultants; (b) stress couples.

2.3 *The boundary conditions*

The boundary conditions at $r = 1$ are:

$$\left.\begin{aligned} N_r^T &= 0 \\ N_{r\theta}^T &= 0 \end{aligned}\right\} \tag{6a}$$

$$\left.\begin{aligned} &M_r = 0 \\ &\int_0^{2\pi} Q_r^* \, r \, \mathrm{d}\theta = \pi r_0^2 p \text{ (for the case of internal pressure)} \\ &\text{and } Q_r^* = 0 \text{ (for the case of tension)} \end{aligned}\right\}. \tag{6b}$$

3. METHOD OF ANALYSIS

The method of analysis involves perturbations with respect to the parameters β and ε. The hole is assumed to be small enough so that $\beta \ll 1$, i.e.,

$$r_0/\sqrt{(R_0 h)} < 1$$

In other words,

$$r_0/R_0 < (h/R_0)^{\frac{1}{2}}$$

or

$$\varepsilon < (h/R_0)^{\frac{1}{2}} \tan \alpha. \tag{7}$$

Hence, even for α as large as 45°, ε will be a small quantity for thin shells for which $h/R_0 \ll 1$. Therefore, the complex stress-displacement function and all the stress resultants can be expressed in series in powers of ε.

$$\begin{aligned}
\phi &= \sum_{j=0}^{\infty} \varepsilon^j \phi_{(j)} \\
N_r &= \sum_{j=0}^{\infty} \varepsilon^j N_{r(j)} \\
N_\theta &= \sum_{j=0}^{\infty} \varepsilon^j N_{\theta(j)} \\
N_{r\theta} &= \sum_{j=0}^{\infty} \varepsilon^j N_{r\theta(j)} \\
M_r &= \sum_{j=0}^{\infty} \varepsilon^j M_{r(j)} \\
M_\theta &= \sum_{j=0}^{\infty} \varepsilon^j M_{\theta(j)}, \text{ etc.}
\end{aligned} \tag{8}$$

Later it will be found that it is necessary to expand each one of the terms in the ε power series further in even powers of β, and products of even powers of β and powers of $\ln \beta$. For instance

$$\phi_{(j)} = \phi_{(j)(0)} + \beta^2 \ln \beta \phi_{j(1)} + \beta^2 \phi_{(j)(2)} + \beta^4 \ln \beta \, . \, \phi_{j(3)} + \dots$$

We use similar expansions for W, F, N_r, N_θ, $N_{r\theta}$, M_r, M_θ, $M_{r\theta}$, Q_r and Q_θ

We now substitute

$$\phi = \phi_{(0)} + \varepsilon \phi_{(1)} + \varepsilon^2 \phi_{(2)} + \dots$$

and

$$s = \frac{1}{\varepsilon} \sqrt{[1 + 2\varepsilon x + \varepsilon^2 (x^2 + y^2)]}$$

in equation (4). Expanding in powers of ε and collecting coefficients of ε^j, we obtain

$$\nabla^4 \phi_{(j)} + 8i\beta^2 \frac{\partial^2 \phi_{(j)}}{\partial x^2} = i\beta^2 \sum_{n=0}^{(j-1)} L_{(j-n)} \phi_{(n)} \tag{9}$$

where $L_{(j-n)}$ is a differential operator. For example,

$$L_{(1)} = 8x\frac{\partial^2}{\partial x^2} - 16y\frac{\partial^2}{\partial x\,.\,\partial y} \tag{10a}$$

$$L_{(2)} = 4(y^2-2x^2)\frac{\partial^2}{\partial x^2} + 32xy\frac{\partial^2}{\partial x\,.\,\partial y} + 8y^2\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right). \tag{10b}$$

4. SOLUTION FOR SMALL VALUES OF ε AND β

4.1

The membrane and bending solutions will be obtained by considering only the first three terms of the series in equations (8), namely

$$\begin{aligned}
\phi &= \phi_{(0)} + \varepsilon\phi_{(1)} + \varepsilon^2\phi_{(2)} + \ldots \\
N_r &= N_{r(0)} + \varepsilon N_{r(1)} + \varepsilon^2 N_{r(2)} + \ldots \\
N_\theta &= N_{\theta(0)} + \varepsilon N_{\theta(1)} + \varepsilon^2 N_\theta(2) + \ldots \\
M_\theta &= M_{\theta(0)} + \varepsilon M_{\theta(1)} + \varepsilon^2 M_{\theta(2)} + \ldots .
\end{aligned}$$

For this approximation, equation (9) reduces to three differential equations for $\phi_{(0)}$, $\phi_{(1)}$ and $\phi_{(2)}$ given by

$$\nabla^4\phi_{(0)} + 8i\beta^2\frac{\partial^2\phi_{(0)}}{\partial x^2} = 0 \tag{11a}$$

$$\nabla^4\phi_{(1)} + 8i\beta^2\frac{\partial^2\phi_{(1)}}{\partial x^2} = i\beta^2 L_{(1)}[\phi_{(0)}] \tag{11b}$$

$$\nabla^4\phi_{(2)} + 8i\beta^2\frac{\partial^2\phi_{(2)}}{\partial x^2} = i\beta^2\{L_{(1)}[\phi_{(1)}] + L_{(2)}[\phi_{(0)}]\}. \tag{11c}$$

The differential operators $L_{(1)}$ and $L_{(2)}$ are defined in equations (10a) and (10b).

Equation (11a) is same as the well-known shallow shell equation for cylindrical shells. In fact, solving this equation with its boundary conditions is actually equivalent to solving the problem of a circular opening in a cylindrical shell, for which a perturbation solution in β has already been obtained by Lurie [2] for both the membrane and bending cases. Therefore, no attempt is made here to solve this equation with its associated boundary conditions. In this paper, we obtain a solution of equation (11b) and (11c) and add it to the corrected Lurie's solution [3] for equation (11a). The solution of each of the equations (11b) and (11c) consists of two parts, namely the complimentary solution and the particular integral. The complimentary solution is similar to that obtained by Lurie [2] and Van dyke [1], except that here we have to consider all the terms which are symmetric with respect to x-axis. This solution can be expressed as a product of Krylov functions, Hankel functions of the first kind and trigonometric functions, as follows:

$$\phi_{(j)} = \sum_{n=0}^{\infty} (A_{n1} + iB_{n1})(\alpha_{n1} + i\beta_{n1}) + \sum_{n=0}^{\infty} (A_{n2} + iB_{n2})(\alpha_{n2} + i\beta_{n2}) \tag{12}$$

where

$$\alpha_{n1}+i\beta_{n1} = \cosh[(1-i)\beta x]H_n^{(1)}[\sqrt{(2i)}\beta r]\cos n\theta$$

$$\alpha_{n2}+i\beta_{n2} = \sinh[(1-i)\beta x]H_n^{(1)}[\sqrt{(2i)}\beta r]\cos n\theta. \tag{13}$$

4.2 *Modified boundary conditions*

We now reformulate our boundary conditions (6a) and (6b) in powers of ε, as follows;

The details of the derivation of basic membrane stress resultants are given in the Appendix. These are obtained as power series in ε. To an accuracy of the order of ε^2, the membrane boundary conditions at $r = 1$ can be formulated as follows:

$$N_{r(j)}+\bar{N}_{r(j)} = 0$$

$$N_{r\theta(j)}+\bar{N}_{r\theta(j)} = 0$$

where $j = 0$, 1 and 2.

Substituting for $N_{r(j)}$ and $N_{r\theta(j)}$ ($j = 1, 2$) from equation (5) and for $\bar{N}_{r(j)}$ and $\bar{N}_{r\theta(j)}$ ($j = 1, 2$) from the Appendix in the above equations, we obtain the following boundary conditions for the first and second order approximation in ε:

$$\frac{1}{r_0^2}\left(\frac{1}{r}\frac{\partial F_{(1)(0)}}{\partial r}+\frac{1}{r^2}\frac{\partial^2 F_{(1)(0)}}{\partial\theta^2}\right) = \frac{2p_0-3q_0}{8}\cos\theta+\frac{6p_0-q_0}{8}\cos 3\theta,$$

$$-\frac{1}{r_0^2}\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial F_{(1)(0)}}{\partial\theta}\right)\right] = \frac{(2p_0-3q_0)}{8}\sin\theta-\frac{(6p_0-q_0)}{8}\sin 3\theta,$$

$$\frac{1}{r}\frac{\partial F_{(1)(j)}}{\partial r}+\frac{1}{r^2}\frac{\partial^2 F_{(1)(j)}}{\partial\theta^2} = 0,$$

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial F_{(1)(j)}}{\partial\theta}\right) = 0$$

where $j = 1$ or 2.

$$\frac{1}{r_0^2}\left(\frac{1}{r}\frac{\partial F_{(2)(0)}}{\partial r}+\frac{1}{r^2}\frac{\partial^2 F_{(2)(0)}}{\partial\theta^2}\right) = \frac{(2p_0+3q_0)}{32}+\frac{(30p_0-3q_0)}{32}\cos 4\theta,$$

$$-\frac{1}{r_0^2}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial F_{(2)(0)}}{\partial\theta}\right) = \frac{(12p_0-6q_0)}{32}\sin 2\theta+\frac{(3q_0-30p_0)}{32}\sin 4\theta,$$

$$\frac{1}{r}\frac{\partial F_{(2)(j)}}{\partial r}+\frac{1}{r^2}\frac{\partial^2 F_{(2)(j)}}{\partial\theta^2} = 0, \tag{14}$$

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial F_{(2)(j)}}{\partial\theta}\right) = 0$$

where $j = 1, 2$.

The boundary conditions for bending solutions at $r = 1$ given by (6b) can be written as

$$
\begin{aligned}
M_{r(0)} &= 0 \\
\int_0^{2\pi} Q^*_{r(0)} r \,\mathrm{d}\theta &= \pi r_0^2 p, \qquad \text{for internal pressure case} \\
Q^*_{r(0)} &= 0 \qquad \text{for axial tension case} \\
M_{r(j)} &= 0 \qquad (j = 1, 2) \\
\int_0^{2\pi} Q^*_{r(j)} r \,\mathrm{d}\theta &= 0 \qquad \text{for internal pressure case } (j = 1, 2) \\
Q^*_{r(j)} &= 0 \qquad \text{for axial tension case } (j = 1, 2).
\end{aligned}
\tag{15}
$$

5. FIRST APPROXIMATION

We now proceed to evaluate $\phi_{(1)}$, i.e. the first approximation in ε.

The particular integral for $\phi_{(1)}$ is obtained by substituting for $\phi_{(0)}$ on the right hand side of equation (11b). The function $\phi_{(0)}$ is of the form

$$\phi_{(0)} = W_{(0)} - imF_{(0)}$$

where

$$
\begin{aligned}
W_{(0)} &= \beta^2 \ln \beta \,.\, W_{(0)(1)} + \beta^2 W_{(0)(2)} \\
&\quad + \beta^4 \ln \beta \,.\, W_{(0)(3)} + \dots \\
F_{(0)} &= F_{(0)(0)} + \beta^2 \ln \beta \,.\, F_{(0)(1)} \\
&\quad + \beta^2 F_{(0)(2)} + \beta^4 \ln \beta \,.\, F_{(0)(3)} + \dots .
\end{aligned}
$$

The equation (11b) becomes

$$\nabla^4 \phi_{(1)} + 8i\beta^2 \frac{\partial^2 \phi_{(1)}}{\partial x^2} = \beta^2 m \left[8x \frac{\partial^2 F_{(0)(0)}}{\partial x^2} - 16y \frac{\partial^2 F_{(0)(0)}}{\partial x \,.\, \partial y} \right] + O(\beta^4 \ln \beta). \tag{16}$$

The particular integral corresponding to this is to be assumed as

$$\phi_{(1)} = \beta^2 \phi_{(1)(2)} + \beta^4 \ln \beta \,.\, \phi_{(1)(3)} + \dots$$

Substituting this in equation (16) and equating coefficients of like powers of β we obtain

$$\nabla^4 \phi_{(1)(2)} = m \left[8x \frac{\partial^2 F_{(0)(0)}}{\partial x^2} - 16y \frac{\partial^2 F_{(0)(0)}}{\partial x \,.\, \partial y} \right] \tag{17}$$

For the sake of convenience, we rewrite equation (17) in polar co-ordinates;

$$
\begin{aligned}
\nabla^4 \phi_{(1)(2)} = m \Bigg[& 2(\cos\theta + 3\cos 3\theta) r \frac{\partial^2 F_{(0)(0)}}{\partial r^2} \\
& + 6(\cos\theta - \cos 3\theta) \left(\frac{\partial F_{(0)(0)}}{\partial r} + \frac{1}{r} \frac{\partial^2 F_{(0)(0)}}{\partial \theta^2} \right) \\
& + 4(3\sin 3\theta - \sin\theta) \left(\frac{1}{r} \frac{\partial F_{(0)(0)}}{\partial \theta} - \frac{\partial^2 F_{(0)(0)}}{\partial r \,.\, \partial \theta} \right) \Bigg].
\end{aligned}
\tag{18}
$$

5.1 *Membrane stresses*

It is evident from equation (16) that the particular integral of $\phi_{(1)}$ to our degree of approximation in β, contains only real part thus contributing to only bending stresses, i.e. the membrane stresses involving ε can be evaluated only from the complimentary solution. The number of terms to be considered in (12) depends upon the desired degree of accuracy, whereas the choice of particular terms depends upon the boundary conditions. For this, one has to examine the expansions for α_{n1}, β_{n1}, α_{n2} and β_{n2} and choose in equation (12) those values of n which can contribute the appropriate terms necessary to satisfy the boundary conditions. The associated arbitrary constants in equation (12) are also taken in the form of series in powers of β and products of powers of β and log β in such a way that all the boundary conditions can be satisfied.

Separating the real and imaginary terms in equation (12), we get the stress function in the form

$$\begin{aligned} F_{(1)} = {} & A_1\frac{\cos\theta}{r}+A_2\frac{\cos 3\theta}{r^3}+A_3\frac{\cos 3\theta}{r} \\ & +\beta^2\ln\beta\left[A_4\frac{\cos\theta}{r}+A_5\frac{\cos 3\theta}{r^3}+A_6\frac{\cos 3\theta}{r}\right] \\ & +\beta^2\left[\frac{\pi(A_3-A_1)}{4}r\cos\theta+A_7\frac{\cos\theta}{r}\right. \\ & \left.+A_8\frac{\cos 3\theta}{r^3}+A_9\frac{\cos 3\theta}{r}\right] \end{aligned}$$

where A_1 to A_9 are independent constants, which can be evaluated from the boundary conditions, equation (14).

$$A_1 = \frac{3q_0-2p_0}{16}r_0^2$$

$$A_2 = \frac{6p_0-q_0}{24}r_0^2$$

$$A_3 = \frac{q_0-6p_0}{16}r_0^2$$

and all the constants A_4 to A_9 vanish individually. The membrane force $N_{\theta(1)}$ is now calculated from

$$\begin{aligned} N^T_{\theta(1)} = \bar{N}_{\theta(1)}+\frac{1}{r_0^2}\frac{\partial^2}{\partial r^2}F_{(1)} = {} & \frac{(9q_0-6p_0)}{8}r\cos\theta \\ & +\frac{(6p_0-q_0)}{8}r\cos 3\theta+\frac{(3q_0-2p_0)}{8}\frac{\cos\theta}{r^3} \\ & -\frac{q_0-6p_0}{2}\frac{\cos 3\theta}{r^5}-\frac{(6p_0-q_0)}{8}\frac{\cos 3\theta}{r^3}. \end{aligned} \tag{19}$$

5.2 *Bending stresses*

The particular integral of $W_{(1)}$ from equation (18) is given by

$$W_{(1)(1)}\,[\text{particular integral}] = 0$$

$$W_{(1)(2)}\,[\text{particular integral}] = mr_0^2\left[\frac{-(6p_0+q_0)}{16}r^3\log r\cos\theta\right.$$

$$+\frac{(2p_0-13q_0)}{48}r^3\log r\cos 3\theta+\frac{(2p_0-q_0)}{64}r^3\cos 5\theta$$

$$\left.+\frac{(6p_0-3q_0)}{64}r\cos 3\theta+\frac{q_0-2p_0}{64}r\cos 5\theta\right].$$

The complimentary solutions for $W_{(1)}$ are given by

$$W_{(1)(1)}\,(\text{complimentary solution}) = mr_0^2\left[C_1-\frac{(2p_0+q_0)}{8}\right]r\cos\theta$$

$$W_{(1)(2)}\,(\text{complimentary solution}) = mr_0^2\left[C_1 r\left(\ln\frac{\gamma r}{\sqrt{2}}\right)\cos\theta\right.$$

$$+\frac{C_2}{r}\cos\theta-\frac{(2p_0+q_0)}{8}r\left(\ln\frac{\gamma r}{\sqrt{2}}\right)\cos\theta$$

$$-\frac{(6p_0+q_0)}{48}r\cos\theta+\frac{(q_0-6p_0)}{96}\frac{\cos\theta}{r}$$

$$+\left(\frac{C_3}{r}+\frac{C_4}{r^3}\right)\cos 3\theta+\frac{(q_0-6p_0)}{96}\frac{\cos 3\theta}{r}$$

$$-\frac{(10p_0+q_0)}{64}r\cos 3\theta+\frac{C_5}{r^3}\cos 5\theta+\frac{C_6}{r^5}\cos 5\theta$$

$$\left.+\frac{(q_0-6p_0)}{96}\frac{1}{r}\cos 5\theta+\frac{(6p_0-q_0)}{192}r\cos 5\theta\right].$$

The complete solution for $W_{(1)}$ is given by

$$W_{(1)} = \beta^2\ln\beta W_{(1)(1)}+\beta^2 W_{(1)(2)}$$

where

$$W_{(1)(j)} = W_{(1)(j)}\,(\text{particular integral})$$

$$+\,W_{(1)(j)}\,(\text{complimentary solution})$$

where $j = 1, 2$

The constants $C_1, C_2 \ldots C_6$ are determined from boundary conditions (15) and are given below:

$$C_1 = -\frac{p_0}{2}$$

$$C_2 = \frac{[6p_0(11+4v)+5q_0(2+v)]}{48(1-v)}$$

$$C_3 = \frac{[p_0(14+8v)+q_0(12v-19)]}{48(3+v)}$$

$$C_4 = \frac{[2p_0(6v^2-12v-10)+q_0(57v^2-84v+115)]}{18(1-v)(3+v)}$$

$$C_5 = \frac{[120p_0+5q_0(v-7)]}{480(3+v)}$$

$$C_6 = \frac{[6p_0(5v-17)+5q_0(7-3v)]}{960(3+v)}.$$

From equations (5), the expression for $M^T_{\theta(1)}$ becomes:

$$\begin{aligned}
M^T_{\theta(1)} = Dm\beta^2\Bigg[& -\cos\theta\Bigg\{\frac{(7+3v)(6p_0+q_0)}{16}\frac{1}{r^3} \\
& +\frac{(6p_0+q_0)(1+v)}{8}\frac{1}{r}+\frac{(6p_0+q_0)}{16} \\
& \times[(2+6v)\log r+(5v+1)]r\Bigg\}-\cos 3\theta \\
& \times\Bigg\{\frac{4p_0(6v^2-12v-10)+2q_0(57v^2-84v+115)}{3(3+v)}\frac{1}{r^5} \\
& +\frac{[10p_0(1+v)+5q_0(5v-7)](5-v)}{48(3+v)}\frac{1}{r^3} \\
& -\frac{(p_0+q_0)}{2}\frac{1}{r}+\frac{(13q_0-2p_0)[(5v+1)-6(1-v)\ln r]}{48}r\Bigg\} \\
& -\cos 5\theta\Bigg\{\frac{[6p_0(5v-17)+5q_0(7-3v)](1-v)}{32(3+v)}\frac{1}{r^7} \\
& +\frac{[24p_0+q_0(v-7)](7-3v)}{24(3+v)}\frac{1}{r^5} \\
& +\frac{(6p_0-q_0)(v-13)}{48}\frac{1}{r^3}+\frac{q_0}{4}\frac{1}{r} \\
& +\frac{(q_0-2p_0)(3v-11)}{32}r\Bigg\}\Bigg].
\end{aligned}$$

6. SECOND APPROXIMATION

In equation (11c), we substitute

$$\phi_{(0)} = W_{(0)} - imF_{(0)}$$

$$\phi_{(1)} = W_{(1)} - imF_{(1)}$$

where $W_{(0)}$ and $F_{(0)}$ are of the form given in Section 5 whereas $W_{(1)}$ and $F_{(1)}$ are of the form

$$W_{(1)} = \beta^2 \ln \beta \,.\, W_{(1)(1)} + \beta^2 W_{(1)(2)} + \ldots$$

$$F_{(1)} = F_{(1)(0)} + \beta^2 \ln \beta \,.\, F_{(1)(1)} + \beta^2 F_{(1)(2)} + \ldots .$$

It then follows from equation (11c) that the particular integral for $\phi_{(2)(2)}$ to our degree of approximation in β contains only real part and hence does not contribute to the membrane solution. We now proceed to obtain the membrane solution in second approximation in ε using only the complimentary solution.

Proceeding along the same lines as given in Section 5.1, we get an expression for the stress function $F_{(2)}$ in the form

$$\begin{aligned} F_2 = {} & B_1 \ln \beta + B_2 + B_3 \ln r + B_4 \cos 2\theta + B_5 \frac{\cos 2\theta}{r^2} \\ & + B_6 \frac{\cos 4\theta}{r^4} + B_7 \frac{\cos 4\theta}{r^2} + \beta^2 \ln \beta \left[B_8 + B_9 \ln r \right. \\ & \left. + B_{10} \cos 2\theta + B_{11} \frac{\cos 2\theta}{r^2} + B_{12} \frac{\cos 4\theta}{r^4} + B_{13} \frac{\cos 4\theta}{r^2} \right] \\ & + \beta^2 \left[B_{14} + B_{15} \ln r - \frac{\pi}{4}(B_4 + B_3) r^2 + B_{16} \cos 2\theta \right. \\ & + B_{17} \frac{\cos 2\theta}{r^2} - \frac{\pi}{8}(2B_4 + B_3) r^2 \cos 2\theta + B_{18} \frac{\cos 4\theta}{r^2} \\ & \left. + B_{19} \frac{\cos 4\theta}{r^4} \right] + \beta^2 (\ln \beta)^2 B_{20} . \end{aligned}$$

We can evaluate the arbitrary constants in the above equation from boundary conditions (14).

$$B_3 = -\frac{(2p_0 + 3q_0) r_0^2}{32}$$

$$B_4 = \frac{(3q_0 - 6p_0) r_0^2}{32}$$

$$B_5 = \frac{(2p_0 - q_0) r_0^2}{16}$$

$$B_6 = \frac{(3q_0 - 30p_0) r_0^2}{128}$$

$$B_7 = \frac{(10p_0 - q_0) r_0^2}{32} .$$

All the constants B_8 to B_{13} vanish individually

$$B_{15} = -\frac{\pi p_0 r_0^2}{8}$$

$$B_{16} = \frac{\pi(3q_0 - 14p_0)r_0^2}{128}$$

$$B_{17} = \frac{\pi(14p_0 - 3q_0)r_0^2}{256}.$$

The constants B_{18} and B_{19} also vanish individually. We can now obtain an expression for the membrane stresses in the ε^2 approximation from the stress function $F_{(2)}$.

$$\begin{aligned} N^T_{\theta(2)} = {} & \frac{(2p_0+3q_0)}{32}\left(3r^2+\frac{1}{r^2}\right)+\frac{(6p_0-3q_0)}{8}\left(r^2+\frac{1}{r^4}\right)\cos 2\theta \\ & +\frac{(3q_0-30p_0)}{32}\left(r^2-\frac{2}{r^4}+\frac{5}{r^6}\right)\cos 4\theta \\ & +\pi\beta^2\left[\frac{p_0}{8}\left(1+\frac{1}{r^2}\right)+\frac{(14p_0-3q_0)}{128}\left(1+\frac{3}{r^4}\right)\cos 2\theta\right]. \end{aligned} \tag{20}$$

The evaluation of bending stresses in the ε^2 approximation involves very tedious algebra. But a careful examination shows that the magnitude of these stresses is very insignificant. The real part of $\phi_{(2)}$ involves β^2 terms and hence the bending stresses in the second approximation include the term $\varepsilon^2\beta^2$. It can be seen from equation (7) that for a cone angle even as large as 90° ($\alpha \approx 45°$), ε is only of the order of $(h/R_0)^{\frac{1}{2}}$ which is quite small for a thin shell. In other words, in practice, ε will be smaller than β itself i.e. $\varepsilon^2\beta^2$ will be smaller than β^4. Since, we have already neglected terms involving $\beta^4 \ln \beta$ everywhere, we are justified in omitting the bending stresses in the ε^2 approximation.

7. COMPLETE SOLUTION

The complete solution to the problem up to terms including ε^2 can be written by adding the zeroth, first and second approximation solutions. Since the only non-vanishing stress along the boundary of the hole is σ_θ, we derive here expressions only for this stress component at $r = 1$.

$$\sigma_\theta = \sigma_{\theta(m)} + \sigma_{\theta(b)}$$

where the subscripts m and b denote membrane and bending solutions.

$$\sigma^T_{\theta(m)} = \frac{1}{h}[N^T_{\theta(0)} + \varepsilon N^T_{\theta(1)} + \varepsilon^2 N^T_{\theta(2)}]$$

$$\sigma^T_{\theta(b)} = \frac{6}{h^2}[M^T_{\theta(0)} + \varepsilon M^T_{\theta(1)}].$$

The expressions for $N^T_{\theta(0)}$ and $M^T_{\theta(0)}$ can be taken from the corrected Lurie's solution. Complete expressions for the membrane and bending components of σ_θ at $r = 1$ are recorded below:

$$
\begin{aligned}
[h\sigma^T_{\theta(m)}]_{r=1} = {} & \frac{(2p_0+3q_0)}{2}+(q_0-2p_0)\cos 2\theta \\
& +\frac{\pi\beta^2}{4}[4q_0+(5q_0-2p_0)\cos 2\theta] \\
& +\frac{\varepsilon}{2}[(3q_0-2p_0)\cos\theta+(6p_0-q_0)\cos 3\theta] \\
& +\frac{\varepsilon^2}{32}[(8p_0+12q_0)+(48p_0-24q_0)\cos 2\theta \\
& +(12q_0-120p_0)\cos 4\theta \\
& +\pi\beta^2\{8p_0+(14p_0-3q_0)\cos 2\theta\}].
\end{aligned}
$$

$$
\begin{aligned}
[h\sigma^T_{\theta(b)}]_{r=1} = {} & \frac{3(1+\nu)\beta^2}{[3(1-\nu^2)]^{\frac{1}{2}}}\left[4q_0\ln\frac{\gamma\beta}{\sqrt{2}}+\frac{9q_0}{4}-\frac{p_0}{2}\right. \\
& +\left\{\frac{q_0}{12}\frac{(19+41\nu)}{(3+\nu)}-\frac{p_0}{6}\frac{(7+5\nu)}{(3+\nu)}\right. \\
& \left.+\ln\frac{\gamma\beta}{\sqrt{2}}(2p_0-5q_0)\frac{(1-\nu)}{(3+\nu)}\right\}\cos 2\theta \\
& \left.+\frac{(2p_0-q_0)}{4}\frac{(1-\nu)}{(3+\nu)}\cos 4\theta\right] \\
& -\frac{3(1+\nu)\beta^2\varepsilon}{[3(1-\nu^2)]^{\frac{1}{2}}}\left[\frac{5(6p_0+q_0)}{8}\cos\theta\right. \\
& +\frac{\cos 3\theta}{24(3+\nu)}\{p_0(2\nu-34)+11q_0(1+7\nu)\} \\
& \left.+\frac{\cos 5\theta}{12(3+\nu)}(1-\nu)(12p_0-5q_0)\right].
\end{aligned}
$$

8. DISCUSSION

It is interesting to note that ε order term in the membrane stress N_θ^T is independent of β as in equation (19). This is due to the fact that the constants A_4 to A_9 in the expression for $F_{(1)}$ vanish individually and also the non-vanishing $r\cos\theta$ term does not contribute any stress in the shell. But, the ε^2 approximation term in $N_\theta^{(T)}$ does depend on β as in equation (20).

The membrane and bending solutions have been obtained to an accuracy of the order of β^2 and ε^2. These have been plotted in Figs. 3–6 for $\beta = 0{\cdot}3$ and $\varepsilon = 0, 0{\cdot}04$ and $0{\cdot}069$. These different values of ε correspond to semi-cone angles of 0°, 30° and 45° for a shell whose R_0/h is 46·2. The Poisson's ratio is assumed to be 0·3. The variation of stresses in the conical shell from the cylindrical shell is observed to be more in the case of axial tension loading than in the case of internal pressure.

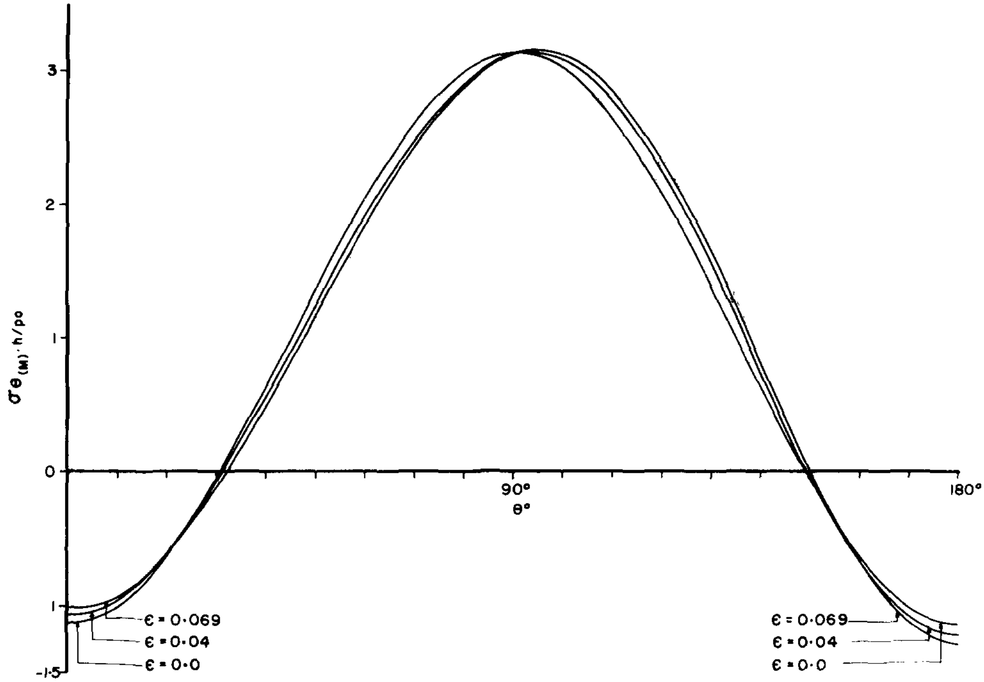

FIG. 3. Membrane stresses due to axial tension (β = 0·3).

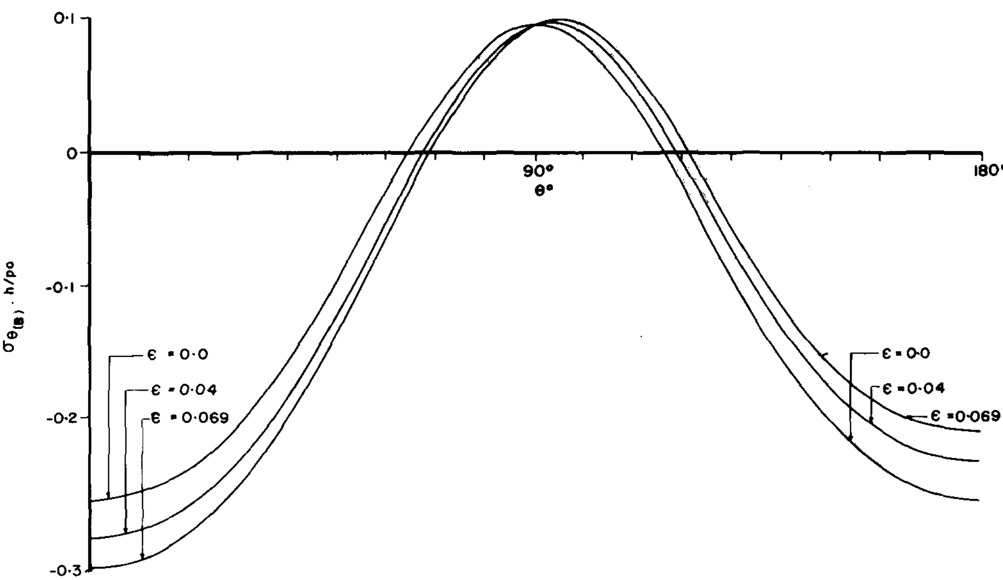

FIG. 4. Bending stresses due to axial tension (β = 0·3).

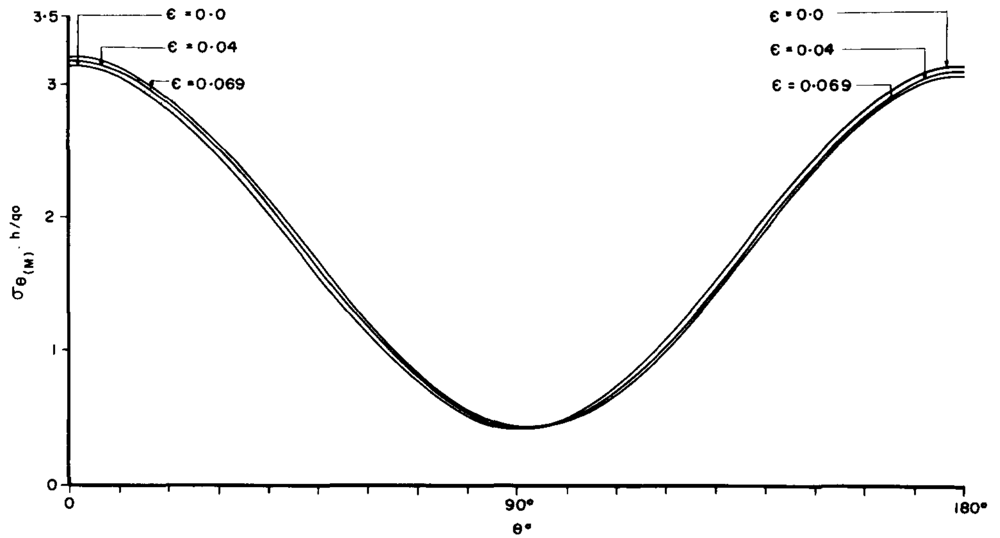

FIG. 5. Membrane stresses due to internal pressure (β = 0·3).

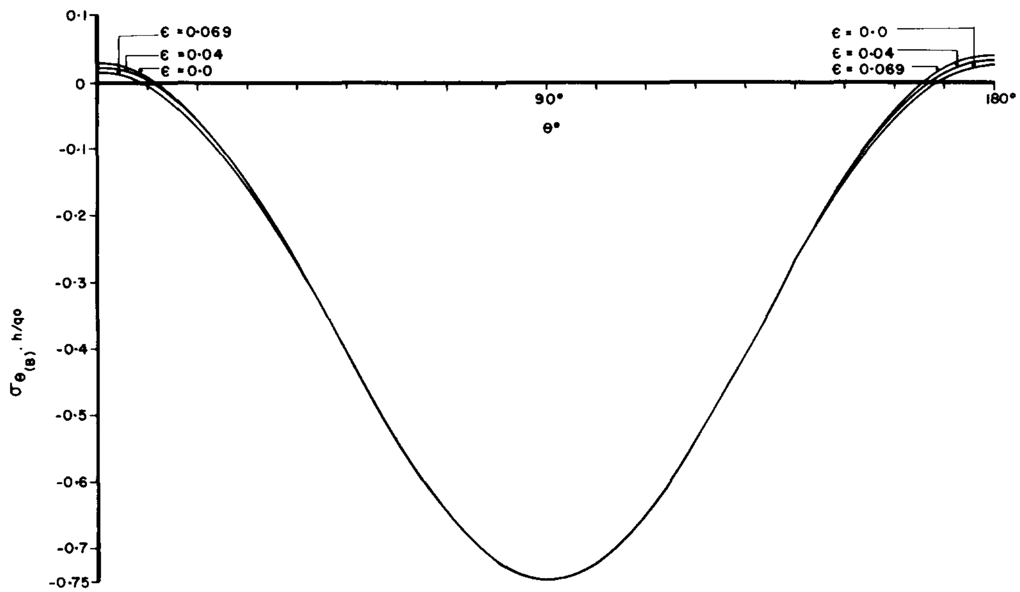

FIG. 6. Bending stresses due to internal pressure (β = 0·3).

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APPENDIX

DERIVATION OF BASIC STRESSES $\bar{N}_r$, $\bar{N}_\theta$ AND $\bar{N}_{r\theta}$

Consider a conical shell as shown in Fig. 1(b) closed at both ends and subjected to a uniform internal pressure p and an axial load of P. Then from equations of equilibrium, it is possible to show that the membrane stresses are given by

$$\begin{aligned} \bar{N}_s &= \frac{p_0}{(1+2\varepsilon r\cos\theta+\varepsilon^2r^2)^{\frac{1}{2}}}+\frac{q_0}{2}(1+2\varepsilon r\cos\theta+\varepsilon^2r^2)^{\frac{1}{2}} \\ \bar{N}_\psi &= q_0(1+2\varepsilon r\cos\theta+\varepsilon^2r^2)^{\frac{1}{2}} \\ \bar{N}_{s\psi} &= 0 \end{aligned} \tag{A1}$$

where

$$q_0 = pR_0,$$

$$p_0 = \frac{P}{2\pi R_0\cos^2\alpha}.$$

We shall now calculate $\bar{N}_r$, $\bar{N}_\theta$ and $\bar{N}_{r\theta}$ from the above stresses by using the transformations

$$\begin{aligned} \bar{N}_r &= \tfrac{1}{2}(\bar{N}_s+\bar{N}_\psi)+\tfrac{1}{2}(\bar{N}_s-\bar{N}_\psi)\cos 2\lambda \\ \bar{N}_\theta &= \tfrac{1}{2}(\bar{N}_s+\bar{N}_\psi)-\tfrac{1}{2}(\bar{N}_s-\bar{N}_\psi)\cos 2\lambda \\ \bar{N}_{r\theta} &= -\tfrac{1}{2}(\bar{N}_s-\bar{N}_\psi)\sin 2\lambda \end{aligned} \tag{A2}$$

where λ is the angle between s and r (see Fig. 1a).

It is possible to show that

$$\begin{aligned} \cos 2\lambda &= \frac{(\cos 2\theta+2\varepsilon r\cos\theta+\varepsilon^2r^2)}{(1+2\varepsilon r\cos\theta+\varepsilon^2r^2)} \\ \sin 2\lambda &= \frac{(\sin 2\theta+2\varepsilon r\sin\theta)}{(1+2\varepsilon r\cos\theta+\varepsilon^2r^2)}. \end{aligned} \tag{A3}$$

By substituting (A1) and (A3) in (A2) and neglecting terms of order ε^3, we get,

$$\begin{aligned} \bar{N}_r = {} & \frac{3q_0+2p_0}{4}+\frac{2p_0-q_0}{4}\cos 2\theta \\ & +\varepsilon r\left\{\frac{(3q_0-2p_0)}{8}\cos\theta+\frac{(q_0-6p_0)}{8}\cos 3\theta\right\} \\ & +\frac{\varepsilon^2r^2}{32}\{(2p_0+3q_0)+(30p_0-3q_0)\cos 4\theta\} \end{aligned}$$

$$
\begin{aligned}
\bar{N}_\theta &= \frac{3q_0+2p_0}{4}+\frac{q_0-2p_0}{4}\cos 2\theta \\
&\quad +\varepsilon r\left\{\frac{(9q_0-6p_0)}{8}\cos\theta+\frac{(6p_0-q_0)}{8}\cos 3\theta\right\} \\
&\quad +\frac{\varepsilon^2 r^2}{32}\{(6p_0+9q_0)+(24p_0-12q_0)\cos 2\theta+(3q_0-30p_0)\cos 4\theta\} \\
\bar{N}_{r\theta} &= \frac{(q_0-2p_0)}{4}\sin 2\theta \\
&\quad +\varepsilon r\left\{\frac{(6p_0-q_0)}{8}\sin 3\theta+\frac{(3q_0-2p_0)}{8}\sin\theta\right\} \\
&\quad +\frac{\varepsilon^2 r^2}{32}\{(12p_0-6q_0)\sin 2\theta+(3q_0-30p_0)\sin 4\theta\}.
\end{aligned}
$$

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Абстракт—Представляются аналитические решения для напряжений в пологих, конических оболочках, обладающих круглым отверстием на боковой поверхности. Оболочка замкнута на обех торцах и нагруженна равномерным осевым растяжением и внутренним давлением, которые уравновершенны только распределенными поперечными силами сдвига на краях отверстия. Метод расчета возводит встепень возмущения параметров, описивающих кривизну и угол конусной оболочки (β и ϵ относительно). Для малых значений этих параметров определяются существенные мембранные и моментные напряжения, с точностью до членлв порядка β^2 и ϵ^2.